Lecture: Kinematics
Lecture: Content

• Forward kinematics differential drive
Lecture: Content

- Forward kinematics differential drive
- Tri-cycle kinematics
Lecture: Content

- Forward kinematics differential drive
- Tri-cycle kinematics
- Car kinematics
Lecture: Content

- Forward kinematics differential drive
- Tri-cycle kinematics
- Car kinematics
- Articulated vehicle kinematics
Lecture: Content

- Forward kinematics differential drive
- Tri-cycle kinematics
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- Articulated vehicle kinematics
- Car with trailer
Lecture: Content

- Forward kinematics differential drive
- Tri-cycle kinematics
- Car kinematics
- Articulated vehicle kinematics
- Car with trailer
- Nonholonomicity
**Lecture:** Content

- Forward kinematics differential drive
- Tri-cycle kinematics
- Car kinematics
- Articulated vehicle kinematics
- Car with trailer
- Nonholonomicity
- Holonomic robots
Lecture: Content

- Forward kinematics differential drive
- Tri-cycle kinematics
- Car kinematics
- Articulated vehicle kinematics
- Car with trailer
- Nonholonomicity
- Holonomic robots
- ICC - Instantaneous Center of Curvature (COR)
Lecture: Sand buggy at high speed

Dynamics becomes important - slip, skidding
Lecture: Bulldozer - in contact

Tracked vehicles has more advanced kinematics due to friction
Lecture: Hägglunds BAE BV206

Hägglunds BV206 in water - river rafting
Lecture: Wheel Kinematic Constraints

- Moves on a horizontal plane
Lecture: Wheel Kinematic Constraints

- Moves on a horizontal plane
- Point contact on wheel
Lecture: Wheel Kinematic Constraints

- Moves on a horizontal plane
- Point contact on wheel
- Wheel will not deform
Lecture: Wheel Kinematic Constraints

- Moves on a horizontal plane
- Point contact on wheel
- Wheel will not deform
- No slip, skid, or sliding
Lecture: Wheel Kinematic Constraints

- Moves on a horizontal plane
- Point contact on wheel
- Wheel will not deform
- No slip, skid, or sliding
- No friction around contact point
Lecture: Wheel Kinematic Constraints

- Moves on a horizontal plane
- Point contact on wheel
- Wheel will not deform
- No slip, skid, or sliding
- No friction around contact point
- Steering axes are orthogonal to surface
Lecture: Wheel Kinematic Constraints

- Moves on a horizontal plane
- Point contact on wheel
- Wheel will not deform
- No slip, skid, or sliding
- No friction around contact point
- Steering axes are orthogonal to surface
- Wheels are mounted on a frame
Lecture: Kinematics differential driven robot

Differential driven robot - a power wheelchair
**Lecture:** Kinematics differential driven robot

\[
\begin{align*}
\nu &= \frac{v_R + v_L}{2} \\
\omega &= \frac{v_R - v_L}{B} \\
R &= B \cdot \frac{(v_R + v_L)}{(v_R - v_L)} \\
\nu &= \omega \cdot R
\end{align*}
\]

Kinematic equations for a differential driven robot

- Baseline between wheels, \( B \)
**Lecture:** Kinematics differential driven robot

\[
\begin{align*}
  v &= \frac{v_R + v_L}{2} \\
  \omega &= \frac{v_R - v_L}{B} \\
  R &= B \cdot \frac{(v_R + v_L)}{(v_R - v_L)} \\
  v &= \omega \cdot R
\end{align*}
\]

Kinematic equations for a differential driven robot

- Baseline between wheels, \( B \)
- Velocity left wheel, \( v_L \)
Lecture: Kinematics differential driven robot

\[ v = \frac{v_R + v_L}{2} \]
\[ \omega = \frac{v_R - v_L}{B} \]
\[ R = B \cdot \frac{(v_R + v_L)}{(v_R - v_L)} \]
\[ v = \omega \cdot R \]

Kinematic equations for a differential driven robot

- Baseline between wheels, \( B \)
- Velocity left wheel, \( v_L \)
- Velocity right wheel, \( v_R \)
Lecture: Kinematics differential driven robot

\[ v = \frac{v_R + v_L}{2} \]
\[ \omega = \frac{v_R - v_L}{B} \]
\[ R = B \cdot \frac{(v_R + v_L)}{(v_R - v_L)} \]
\[ v = \omega \cdot R \]

Kinematic equations for a differential driven robot

- Baseline between wheels, \( B \)
- Velocity left wheel, \( v_L \)
- Velocity right wheel, \( v_R \)
- Vehicle angular velocity, \( \omega \)
**Lecture:** Kinematics differential driven robot

\[ v = \frac{v_R + v_L}{2} \]
\[ \omega = \frac{v_R - v_L}{B} \]
\[ R = B \cdot \frac{(v_R + v_L)}{(v_R - v_L)} \]
\[ v = \omega \cdot R \]

Kinematic equations for a differential driven robot

- Baseline between wheels, \( B \)
- Velocity left wheel, \( v_L \)
- Velocity right wheel, \( v_R \)
- Vehicle angular velocity, \( \omega \)
- Vehicle forward velocity, \( v \)
Lecture: Kinematics differential driven robot

\[ v = \frac{v_R + v_L}{2} \]
\[ \omega = \frac{v_R - v_L}{B} \]
\[ R = B \cdot \frac{(v_R + v_L)}{(v_R - v_L)} \]
\[ v = \omega \cdot R \]

Kinematic equations for a differential driven robot

- Baseline between wheels, \( B \)
- Velocity left wheel, \( v_L \)
- Velocity right wheel, \( v_R \)
- Vehicle angular velocity, \( \omega \)
- Vehicle forward velocity, \( v \)
- Vehicle turn radius, \( R \) (can be negative)
Lecture: Time Continuous Model - wheelchair

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix}
= \begin{bmatrix}
\cos(\theta) & 0 \\
\sin(\theta) & 0 \\
0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
v \\
\omega
\end{bmatrix}
\]

Time continuous kinematic model

- Differential driven robot (wheelchair)
Lecture: Time Continous Model - wheelchair

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix}
= 
\begin{bmatrix}
\cos(\theta) & 0 \\
\sin(\theta) & 0 \\
0 & 1
\end{bmatrix} 
\cdot 
\begin{bmatrix}
v \\
\omega
\end{bmatrix}
\]

Time continous kinematic model

- Differential driven robot (wheelchair)
- Vehicle orientation, \( \theta \)
Lecture: Time Continuous Model - wheelchair

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix}_G =
\begin{bmatrix}
\cos(\theta) & 0 \\
\sin(\theta) & 0 \\
0 & 1
\end{bmatrix} \cdot 
\begin{bmatrix}
v \\
\omega
\end{bmatrix}
\]

Time continuous kinematic model

- Differential driven robot (wheelchair)
- Vehicle orientation, $\theta$
- Vehicle position, $(x, y)$
Lecture: Time Continuous Model - wheelchair

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix}_G =
\begin{bmatrix}
\cos(\theta) & 0 \\
\sin(\theta) & 0 \\
0 & 1
\end{bmatrix} \cdot
\begin{bmatrix}
v \\
\omega
\end{bmatrix}
\]

Time continuous kinematic model

- Differential driven robot (wheelchair)
- Vehicle orientation, \( \theta \)
- Vehicle position, \((x, y)\)
- Vehicle change in position, \((\dot{x}, \dot{y})\)
Lecture: Time Continuous Model - wheelchair

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix}_G =
\begin{bmatrix}
\cos(\theta) & 0 \\
\sin(\theta) & 0 \\
0 & 1
\end{bmatrix} \cdot 
\begin{bmatrix}
v \\
\omega
\end{bmatrix}
\]

Time continuous kinematic model

- Differential driven robot (wheelchair)
- Vehicle orientation, \( \theta \)
- Vehicle position, \((x, y)\)
- Vehicle change in position, \((\dot{x}, \dot{y})\)
- Forward velocity, \(v\)
Lecture: Time Continous Model - wheelchair

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix}_G =
\begin{bmatrix}
\cos(\theta) & 0 \\
\sin(\theta) & 0 \\
0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
v \\
\omega
\end{bmatrix}
\]

Time continuous kinematic model

- Differential driven robot (wheelchair)
- Vehicle orientation, \( \theta \)
- Vehicle position, \((x, y)\)
- Vehicle change in position, \((\dot{x}, \dot{y})\)
- Forward velocity, \(v\)
- Turning rate, \(\omega\)
Lecture: Tricycle model

A typical tricycle
Lecture: Tricycle model

Tricycle with parameters marked
Lecture: Time continuous tricycle model

\[
\begin{align*}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= \omega \\
v &= v_s \cos(\alpha) \\
\omega &= \frac{v_s \sin(\alpha)}{L}
\end{align*}
\]

Time continuous vehicle model

- $v_s$ - velocity on steering wheel
Lecture: Time continuous tricycle model

\[
\begin{align*}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= \omega \\
v &= v_s \cos(\alpha) \\
\omega &= \frac{v_s \sin(\alpha)}{L}
\end{align*}
\]

Time continuous vehicle model

- $v_s$ - velocity on steering wheel
- $v$ - forward velocity
Lecture: Time continuous tricycle model

\[
\begin{align*}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= \omega \\
v &= v_s \cos(\alpha) \\
\omega &= \frac{v_s \sin(\alpha)}{L}
\end{align*}
\]

Time continuous vehicle model

- $v_s$ - velocity on steering wheel
- $v$ - forward velocity
- $(x, y)$ - position of vehicle in global coordinate
Lecture: Time continuous tricycle model

\[
\begin{align*}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= \omega \\
v &= v_s \cos(\alpha) \\
\omega &= \frac{v_s \sin(\alpha)}{L}
\end{align*}
\]

Time continuous vehicle model

- \( v_s \) - velocity on steering wheel
- \( v \) - forward velocity
- \((x, y)\) - position of vehicle in global coordinate
- \( \alpha \) - steering angle
Lecture: Car model

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta} \\
\dot{\alpha}
\end{bmatrix} = \begin{bmatrix}
\cos(\theta) \\
\sin(\theta) \\
\frac{\tan(\alpha)}{L} \\
0
\end{bmatrix} v + \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} \omega_s
\]

Kinematic model for a car

- Vehicle model with steering wheel, steering angle \( \alpha \)
Lecture: Car model

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta} \\
\dot{\alpha}
\end{bmatrix} =
\begin{bmatrix}
\cos(\theta) \\
\sin(\theta) \\
\tan(\alpha) \\
\frac{L}{v}
\end{bmatrix} v +
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} \omega_s
\]

Kinematic model for a car

- Vehicle model with steering wheel, steering angle $\alpha$
- $(x, y, \theta)$ is the pose of the vehicle (orientation)
Lecture: Car model

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta} \\
\dot{\alpha}
\end{bmatrix} = \begin{bmatrix}
\cos(\theta) \\
\sin(\theta) \\
\frac{\tan(\alpha)}{L} \\
0
\end{bmatrix} v + \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} \omega_s
\]

Kinematic model for a car

- Vehicle model with steering wheel, steering angle $\alpha$
- $(x, y, \theta)$ is the pose of the vehicle (orientation)
- $v$ is the vehicle forward velocity
Lecture: Car model

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta} \\
\dot{\alpha}
\end{bmatrix} =
\begin{bmatrix}
\cos(\theta) \\
\sin(\theta) \\
\frac{\tan(\alpha)}{L} \\
0
\end{bmatrix} v +
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} \omega_s
\]

Kinematic model for a car

- Vehicle model with steering wheel, steering angle $\alpha$
- $(x, y, \theta)$ is the pose of the vehicle (orientation)
- $v$ is the vehicle forward velocity
- $\omega_s$ - angular velocity direction of front wheel
Lecture: Car model

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta} \\
\dot{\alpha}
\end{bmatrix} = \begin{bmatrix}
\cos(\theta) \\
\sin(\theta) \\
\tan(\alpha) \\
\frac{L}{0}
\end{bmatrix} v + \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} \omega_s
\]

Kinematic model for a car

- Vehicle model with steering wheel, steering angle $\alpha$
- $(x, y, \theta)$ is the pose of the vehicle (orientation)
- $v$ is the vehicle forward velocity
- $\omega_s$ - angular velocity direction of front wheel
- Vehicle baseline, $L$
**Lecture: Car model**

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta} \\
\dot{\alpha}
\end{bmatrix} = \begin{bmatrix}
\cos(\theta) \\
\sin(\theta) \\
\tan(\alpha) \\
\frac{L}{0}
\end{bmatrix} v + \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} \omega_s
\]

**Kinematic model for a car**

- Vehicle model with steering wheel, steering angle \( \alpha \)
- \((x, y, \theta)\) is the pose of the vehicle (orientation)
- \(v\) is the vehicle forward velocity
- \(\omega_s\) - angular velocity direction of front wheel
- Vehicle baseline, \(L\)
- COR, \([x_c, y_c]^T = R[-\sin \theta, \cos \theta]^T + [x, y]^T\)
Lecture: Car model

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta} \\
\dot{\alpha}
\end{bmatrix} =
\begin{bmatrix}
\cos(\theta) \\
\sin(\theta) \\
\tan(\alpha) \\
\frac{L}{\tan(\alpha)}
\end{bmatrix} v +
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} \omega_s
\]

Kinematic model for a car

- Vehicle model with steering wheel, steering angle \( \alpha \)
- \((x, y, \theta)\) is the pose of the vehicle (orientation)
- \(v\) is the vehicle forward velocity
- \(\omega_s\) - angular velocity direction of front wheel
- Vehicle baseline, \(L\)
- COR, \([x_c, y_c]^T = R[-\sin \theta, \cos \theta]^T + [x, y]^T\)
- Turn radius, \(R = \frac{L}{\sin(\alpha)}\)
Lecture: Time discrete movement

\[
\begin{bmatrix}
\Delta x_k \\
\Delta y_k \\
\Delta \theta_k
\end{bmatrix} = \left( \frac{2v}{\omega} \right) \sin \left( \frac{\omega T}{2} \right) \begin{bmatrix}
\cos \left( \theta_k + \frac{\omega T}{2} \right) \\
\sin \left( \theta_k + \frac{\omega T}{2} \right) \\
\vdots
\end{bmatrix} \approx T \begin{bmatrix}
v \cos \left( \theta_k + \frac{\omega T}{2} \right) \\
v \sin \left( \theta_k + \frac{\omega T}{2} \right) \\
\omega_k
\end{bmatrix}
\]

Time discrete movement

- State Vector
Lecture: Time discrete movement

\[
\begin{bmatrix}
\Delta x_k \\
\Delta y_k \\
\Delta \theta_k
\end{bmatrix}
= \left( \frac{2v}{\omega} \right) \sin \left( \frac{\omega T}{2} \right)
\begin{bmatrix}
\cos \left( \theta_k + \frac{\omega T}{2} \right) \\
\sin \left( \theta_k + \frac{\omega T}{2} \right) \\
\vdots
\end{bmatrix}
\approx T
\begin{bmatrix}
v \cos \left( \theta_k + \frac{\omega}{2} \right) \\
v \sin \left( \theta_k + \frac{\omega}{2} \right) \\
\omega_k
\end{bmatrix}
\]

Time discrete movement

- State Vector
**Lecture:** Time discrete movement

\[
\begin{bmatrix}
\Delta x_k \\
\Delta y_k \\
\Delta \theta_k
\end{bmatrix} = \left( \frac{2v}{\omega} \right) \sin \left( \frac{\omega T}{2} \right) \begin{bmatrix}
\cos \left( \theta_k + \frac{\omega T}{2} \right) \\
\sin \left( \theta_k + \frac{\omega T}{2} \right) \\
\vdots
\end{bmatrix} \approx T \begin{bmatrix}
v \cos \left( \theta_k + \frac{\omega k}{2} \right) \\
v \sin \left( \theta_k + \frac{\omega k}{2} \right) \\
\omega_k
\end{bmatrix}
\]

Time discrete movement

- State Vector
Lecture: Time discrete movement

\[
\begin{bmatrix}
\Delta x_k \\
\Delta y_k \\
\Delta \theta_k
\end{bmatrix} = \left( \frac{2v}{\omega} \right) \sin \left( \frac{\omega T}{2} \right) \begin{bmatrix}
\cos \left( \theta_k + \frac{\omega T}{2} \right) \\
\sin \left( \theta_k + \frac{\omega T}{2} \right) \\
\vdots
\end{bmatrix} \approx T \begin{bmatrix}
v \cos \left( \theta_k + \frac{\omega}{2} \right) \\
v \sin \left( \theta_k + \frac{\omega}{2} \right) \\
\omega_k
\end{bmatrix}
\]

Time discrete movement

- State Vector
Lecture: Ackermann steering

- Four wheeled vehicle
Lecture: Ackermann steering

- Four wheeled vehicle
- Steering wheels have different angles
Lecture: Ackermann steering

- Four wheeled vehicle
- Steering wheels have different angles
- Four different turning radii
Lecture: Ackermann steering

- Four wheeled vehicle
- Steering wheels have different angles
- Four different turning radii
- Center of rotation is the same (intersection point)
Lecture: Controller

Controller for a mobile robot

• Objective to follow a path or a velocity
Lecture: Controller

Controller for a mobile robot

- Objective to follow a path or a velocity
- Not straightforward non-holonomic system
Controller for a mobile robot

- Objective to follow a path or a velocity
- Not straightforward non-holonomic system
- Kinematics controller does not include dynamics
Controller for a mobile robot

- Objective to follow a path or a velocity
- Not straightforward non-holonomic system
- Kinematics controller does not include dynamics
- Arrive at specific position or pose
Lecture: Car with trailer model

Figure 4. The two centres of rotation for a vehicle with a trailer. The similarity with an articulated vehicle are seen by removing the front wheel in this figure.
Lecture: Car with trailer

Figure 2 - Geometry and Kinematic model of Truck with one trailer

In this theoretical part of the project, the first step was to interpret the kinematic equations dominant on the system. Considering a single trailer is sufficient to start with and yields to the following nonlinear set of differential equations in matrix form as mentioned before.

\[ \text{state} = (x_1, y_1, \theta_1, \phi_1) \text{ in trailer coordinates} \]

\[
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{y}_1 \\
    \dot{\theta}_1 \\
    \dot{\phi}_1
\end{bmatrix} =
\begin{bmatrix}
    v_1 \cos \theta_1 \\
    v_1 \sin \theta_1 \\
    \omega_1 \\
    \omega_c - \omega_1
\end{bmatrix} =
\begin{bmatrix}
    \frac{v_c}{b_c} \sin \varphi_1 - \omega_c \cdot \frac{a_c}{b_c} \cos \varphi_1 \\
    \frac{v_c}{b_c} \sin \varphi_1 + \omega_c (1 + \frac{a_c}{b_c} \cos \varphi_1)
\end{bmatrix}
\]

- \( a_c \): Distance between connection point of tractor and tractor head of trailer
- \( a_1 \): Distance between head of second trailer and first trailer center of back wheels
- \( b_c \): Distance between steering wheel and center of rear wheels of tractor

\[ \omega_c = \frac{v_c}{b_c} \cdot \tan \alpha \]
Lecture: Virtual Steering Wheel

- Useful to put out virtual steering wheel, reversing

Virtual steering wheel - Ake We.

\[
\omega_c = \frac{v_c}{b_c} \cdot \tan \alpha
\]
Lecture: Virtual Steering Wheel

Virtual steering wheel - Ake We.

- Useful to put out virtual steering wheel, reversing
- New kinematic model needs to be created
Lecture: Virtual Steering Wheel

Virtual steering wheel - Ake We.

- Useful to put out virtual steering wheel, reversing
- New kinematic model needs to be created
- Virtual wheel can be placed arbitrary
Virtual steering wheel - Ake We.

- Useful to put out virtual steering wheel, reversing
- New kinematic model needs to be created
- Virtual wheel can be placed arbitrary
- Useful for dog rabbit control